Course Description:

This course provides an in-depth exploration of group theory, a branch of mathematics that studies the properties and structure of groups. Beginning with an introduction to binary operations and the fundamental concepts of groups, the course delves into important examples of groups, such as cyclic groups and Abelian groups, and their properties. Students will also study group actions, including Lagrange's theorem and the orbit-stabilizer theorem, as well as the Burnside-Polya enumeration theorem, which provides a powerful tool for counting objects with symmetry. In addition, the course covers topics such as group presentations, homomorphisms, and the classification of groups, including simple groups and semidirect products. By the end of the course, students will have developed a strong understanding of group theory and its applications to a wide range of mathematical fields, including algebra, geometry, and topology.

Prerequisite:

AP Calculus BC (recommended)

Units:

Unit 1 (Groups): This unit lays the foundation for group theory by introducing the notion of binary operations and Cayley tables. We then define groups as sets with operations that satisfy certain axioms, including the existence of left and right inverses and an identity element. We explore symmetry, isometries, and orientation-preserving isometries as examples of groups, and we investigate the concept of order within a group.

Unit 2 (Examples of Groups): In this unit, we examine a variety of examples of groups, starting with abelian and cyclic groups. We then explore the Klein group, dihedral groups, and duals, which are used to describe symmetries of platonic solids. We also investigate the symmetric groups, which describe permutations of objects, and the quaternion group, which is used in 3D graphics.

Unit 3 (Cyclic Groups): This unit delves deeper into cyclic groups, which are groups generated by a single element. We investigate isomorphisms between cyclic groups and other groups, as well as the concept of subgroups and the generator of a group. We also examine Bezout's Lemma and the fundamental theorem of cyclic groups, which describes the structure of finite cyclic groups.

Unit 4 (Abelian Groups): Abelian groups are groups in which the operation is commutative. We explore product groups and the Chinese remainder theorem for groups, which allows us to combine groups in interesting ways. We also investigate finitely generated abelian groups and the empty group product.

Unit 5 (Group Actions): Group actions are a way of describing how a group acts on a set. We investigate left and trivial actions, orbits, and Lagrange's theorem, which gives us information about the size of a subgroup. We also explore conjugation and the conjugacy class equation, which describes the sizes of conjugacy classes, as well as the orbit-stabilizer theorem, which relates the size of an orbit to the size of its stabilizer. Finally, we examine automorphisms and inner automorphisms, which are important for understanding the structure of a group.

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Unit 6 (Burnside and Polya): In this unit, we explore the Burnside Lemma, which gives a way of counting the number of orbits under a group action. We then investigate Polya's Enumeration Theorem, which applies Burnside's Lemma to counting problems involving coloring and other combinatorial structures. We also explore cycle types and cycle index, which are used in counting problems involving permutations, and we examine class functions, which are important for understanding group representations.

Unit 7 (Quotients): Quotients are a way of constructing new groups from old ones by identifying certain elements as equivalent. We investigate equivalence relations, left and right cosets, and the concept of a normal subgroup, which is a subgroup that is invariant under conjugation. We also explore closure, the index of subgroups, torsion subgroups, and Poincare's Theorem, which relates the torsion subgroups of a group to its cohomology.

Unit 8 (Functions from Groups to Groups): In this unit, we investigate functions from groups to groups, including homomorphisms, injections, and surjections. We explore the domain, codomain, image, range, and kernel of such functions, and we examine the first isomorphism theorem, which relates the kernel of a homomorphism to the image of a group. We also investigate Cayley's theorem, which shows that every group can be embedded in a permutation group, and we examine the second and third isomorphism theorems, which give information about subgroups of a group. Finally, we explore the correspondence theorem, which relates subgroups of a group to subgroups of a quotient group.

Unit 9 (Group Presentations): In this unit, we will explore group presentations, which are a way to describe groups in terms of generators and relations. We will look at how to construct presentations for various groups, including finitely presented and finitely generated groups. We will also study the fundamental theorem of finitely generated abelian groups, which gives a classification of such groups.

Unit 10 (Symmetric Groups): This unit will focus on the symmetric group, which is the group of permutations of a set. We will study the structure of the symmetric group, including its conjugacy classes, 3-cycles, and chirality. We will also examine the relationship between the symmetric group and other groups, including alternating groups and permutation groups.

Unit 11 (Group Classification Theory): In this unit, we will explore the classification of groups, including simple groups, Sylow p-subgroups, and the Sylow theorems. We will also examine normalizers and semidirect products, including twisted and external semidirect products. We will use these tools to classify groups up to isomorphism, including finite groups and certain infinite groups.

Textbook:

D. Jeremy Copeland , *Groups and Fields*, Art of Problem Solving, Fourth printing, 2021

Carter, Nathan, *Visual Group Theory*, AMS / MAA, 2009

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